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Analytic Structure of Gauge Fields

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The analytic structure of gauge fields in the presence of fermions is studied in arbitrary symmetry. A Hamiltonian formalism is developed which relates Cauchy-Riemann equations to the symmetry. The formalism is applied to three problems in $(2+1)$-dimensional Euclidean space: (1) a free fermion, (2) a fermion interacting with a massless scalar field, and (3) a fermion interacting with a vector field. We find that the Hamiltonian for the free fermion is analytic and single-valued in a finite region of momentum space. With the addition of an auxiliary field, the Hamiltonian can be analytic in the entire momentum space. The scalar field then acquires spin-dependent coordinates by interaction with the fermion; the interactions break the Abelian symmetry of ϕ so that $\phi \to \phi_l \sim 1/(x_l - im_l^{-1}(x_l - im_l^{-1})$, where m_l are spin-dependent and multivalued. There are four solutions for each chirality eigenvalue of the fermion. For spinless fermions ϕ gives the Jackiw-Nohl-Rebbi solution and is separable into Coulomb-like $1/x$ analytic functions on the first and fourth quadrants. For a vector field the results are similar except that the coordinates are not spin-dependent or multivalued; interactions break the initial symmetry and $A_\mu(x_\mu) \to A_\mu^l(x_\mu)$ and the A_μ^l have a non-Abelian algebra. The l indices represent directions fixed by spin matrices in a spin-dependent color space.

1. INTRODUCTION

It has been known for many years that a multivalued phase factor is a representation of electromagnetism (Wu and Yang, 1974; Bialnicki-Burula, 1963). The concept follows from the integral of the differential equation representing parallel displacement of a vector around a path through a point x_μ and is an elementary result of differential geometry (Ward, 1977; Ezawa and Tze, 1975). In recent years this concept has been generalized by Yang (1974) and others in studies on the nature of gauge transformations. From the requirement that phase factors in local overlapping regions be single-valued, Wu and Yang obtained several well-known results, including the Dirac quantization of magnetic charge.

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The multivaluedness of the phase factor in gauge field theory follows from the path dependence of an integral. In a complex variable representation the paths are defined on nonanalytic regions. On the other hand, a scalar field has well-known analytic regions, for example, the Coulomb potential for electromagnetism and the Jackiw–Nohl–Rabbi and 't Hooft self-dual gauge potentials (Jackiw et al., 1977; G. 't hooft, unpublished), which are solutions of the Laplace equation. Similar regions exist for vector potentials. More generally, the patching together of local field requires that the fields be analytic in the overlap regions. Thus, although gauge potentials may be predominantly nonanalytic, there nevertheless exist analytic regions. The analytic structure of solutions of the Yang–Mills equations has been considered by several authors (Well, 1979; Penrose and MacCallum, 1972). In other contexts the delineation of analytic regions is important for the integration of Green's functions and location of the mass eigenvalues (poles) in renormalized theories.

In this paper I consider a gauge potential to which is added a fermion and obtain the conditions for the existence of an analytic region of the Hamiltonian. As is well known, the supersymmetric vacuum Hamiltonian vanishes in unbroken symmetry (Iliopoulous and Zumino, 1974; Witten, 1981), as the fermion and boson contributions cancel exactly. It is easy to show that the boson symmetry is the fermion symmetry. Our basic result is that in the Hamiltonian vacuum a scalar (Higgs) potential ϕ acquires the fermion symmetry and spin-dependent coordinates by interaction with the fermion; the interactions break the ϕ symmetry and $\phi \to \phi_l$, where $l = 1, \ldots, 8$, and there are four solutions for each chirality eigenvalue of the fermion. The ϕ_l algebra is chiral $O(2,1) \times T_1 + O(2,1) \times T_1$, where T_1 is in the Cartan subgroup. There are possible $1/r^2$ type singularities, but they are not removable by a gauge transformation, because of the spin components. On the other hand, for a spinless fermion, ϕ gives the Jackiw–Nohl–Rebbi solution and is separable into analytic regions on the first and fourth quadrants of the complex x_μ planes into Coulomb-like $1/x$ potentials. These results help to clarify the analytic structure of the scalar potential and the nature and number of Higgs fields and their internal symmetry (O'Raifeartaigh, 1977) in supersymmetric models (see Section 4).

If ϕ is replaced by a vector potential A_μ, we find that A_μ also acquires (fermionic) spin components by interaction with the fermion in the vacuum, but they appear in exponential terms rather than in the denominator, which in this case has simple zeros. The interactions break the A_μ symmetry and $A_\mu \to A^l_\mu$, where there are four solutions for each chirality eigenvalue. The A^l_ν algebra is non-Abelian. The l indices represent directions fixed by spin matrices in a spin-dependent color (covector) space. For a spinless fermion $A^l_\mu \to A_\mu$, which has analytic regions similar to the Jackiw–Nohl–Reboi solution for ϕ and is separable into $A \sim 1/x$ regions on the first and fourth quadrants. These results help to clarify the analytic structure of the vector potential as well as the physics of the color degrees of freedom.

The organization of the paper is as follows: the formal machinery is assembled in Section 2 and demonstrated for a free fermion in Section 3. A scalar field is added in Section 4 and a vector field in Section 5.

2. ANALYTICITY, SYMMETRY, AND THE VACUUM STATE

It is well known that complex variable methods sometimes offer simple and elegant solutions of problems in integration, topology, function theory, and mathematical physics. Similar advantages exist for differential equations. In this case the complex variable representation is attractive because many of the field equations of motion of physics can be expressed as Cauchy–Riemann equations by a transformation of the background geometry and the fields are defined in an analytic domain. The analytic regions correspond to conserved regions in a real variable representation, as is well known from electrostatics and electrodynamics (e.g., Morse and Feshbach, 1953), and ground-state operator and eigenvalues correspond to points in an analytic region. Here I consider the relation between the analytic regions of a Hamiltonian (operator) and symmetry and show that in the vacuum the Hamiltonian symmetry generates Cauchy–Riemann equations (and the converse) and their solutions define vacuum states (Anderson, 1970). In order to clarify the formalism, I first obtain a relation between the symmetry and the vacuum Hamiltonian for a real variable representation and then extended the result to the complex domain.

The Hamiltonian symmetry is given by the algebra L^{ζ}, where the operators ζ_ν commute with the Hamiltonian,

$$[\zeta_\nu, H(\partial_\mu, x_\mu)] = 0 \tag{1}$$

In this case the eigenvectors of H are degenerate in every multiplet. We look for a transformation that generates the lhs of (1) and is such that H is the vacuum Hamiltonian. Then (1) yields the vacuum symmetry. Consider the continuous transformation

$$T(\alpha_\mu)\psi(x_\mu) = \psi'(x_\mu) \tag{2}$$

where $T(\alpha_\mu)$ are transformation matrices and ψ are vectors in the representation space of the transformation group. Then

$$T(\alpha_\mu) \to T(\alpha_\mu, \zeta_\mu) = \exp(\alpha^\mu \zeta_\mu) \tag{3}$$

where ζ_ν are generators defined in vector space and $\alpha_\mu(x_\mu)$ are parameters defined in convector space and

$$\langle \zeta_\mu, \alpha_\mu \rangle = \text{const} = \varepsilon \tag{4}$$

With ζ_μ in matrix representation, then $\alpha_\mu = \tau^\nu_\mu \beta_\nu$ are matrices and $\beta_\nu \sim \delta x_\nu$ and $\partial\delta x_\nu/\partial x_\mu = \text{const}$. For the operator $H'(\partial_\mu, x_\mu)$ we obtain

$$H'(\partial_\mu, x_\mu) = T(\alpha_\nu, \zeta_\nu) H(\partial_\mu, x_\mu) T^{-1}(\alpha_\nu, \zeta_\nu) \tag{5}$$

Equation (5) gives

$$\begin{aligned} \frac{\partial H'(\zeta_\mu, x_\mu)}{\partial \alpha_\nu} &= u_{\nu\mu} \frac{\partial H'(\partial_\mu, x_\mu)}{\partial x_\mu} \\ &= [\zeta_\nu, H'(\partial_\mu, x_\mu)] + u_{\nu\mu} \frac{\partial H'(\partial_\mu, x_\mu)}{\partial x_\mu} \end{aligned} \tag{6}$$

where $u_{\nu\mu} = dx_\mu / d\alpha_\nu$ are constant matrix elements. Equation (6) follows from the chain rule for differentiation with several complex variables (Bochner and Martin, 1948). Equation (6) can be simplified by means of

$$\zeta_\nu = \frac{\partial}{\partial \alpha_\nu} - \frac{\partial}{\partial \alpha_\nu} + H' \zeta_\nu H'^{-1} \tag{7}$$

and

$$\frac{\partial \alpha_\nu}{\partial \alpha_\mu} = [\zeta_\mu, \alpha_\nu] + \delta_{\mu\nu} \tag{8}$$

which gives the "quantization rules"

$$[\zeta_\mu, \alpha_\nu] = \varepsilon_{\mu\nu}, \qquad \partial\zeta_\nu/\partial\alpha_\mu = 0 \tag{9}$$

where $\varepsilon_{\mu\nu}$ are scale constants, which follow from the integration of (8). Then (6) reduces to

$$\hat{u}_{\nu\mu} \frac{\partial H'(\zeta_\mu, x_\mu)}{\partial x_\mu} = [\zeta_\nu, H'(\zeta_\mu, x_\mu)] \tag{10}$$

where $\hat{u} = u \times \text{const}$. Now $(10) = 0$ equates the vacuum requirement $\partial H'/\partial x_\mu = 0$ to the symmetry constraint $[\zeta_\mu, H'] = 0$, so that the solution of $(10) = 0$ gives the vacuum symmetry defined by L^ζ. Summarizing, we have found a transformation (5) that yields an equation $(10) = 0$ that states that if the vacuum Hamiltonian exists, then it has the symmetry defined by L^ζ.

Now consider the Dirac Hamiltonian $D[\partial_\mu, \phi(x_\mu)]$. If we write out the Dirac matrix, it is evident D is a function of operators and variables in complex domains. In this case the vacuum defined by $\partial H'/\partial x_\mu = 0$ is not single-valued unless $D'(\zeta_\mu, x_\mu)$ is analytic. Then equation (10) has to be replaced by

$$\tilde{u}_{\nu\mu} \frac{\partial D'(\zeta_\mu, x_\mu)}{\partial x_\mu} = [\zeta_\nu, D'(\zeta_\mu, x_\mu)] \tag{11}$$

where

$$D'(\zeta_\mu, x_\mu) = \exp(\alpha^\nu \zeta_\nu) D(\partial_\mu, x_\mu) \exp(-\alpha^\nu \zeta_\nu) \tag{12}$$

and $\tilde{u}_{\nu\mu} = dx_\mu / d\alpha^\nu$ and $\alpha^\mu = \alpha^\mu(x_\mu)$. Equation (11) also can be obtained from the complex representation of the Hamiltonian equations of motion and from the Poisson bracket representation of Hamiltonian dynamics (Anderson, 1970). It is of interest to note that the adjoint operator in the quantization rule (9) has been obtained in the form $z_\mu^\dagger \to -\partial/\partial z^\mu$ in several different studies, by Bargmann (1962) on complex rotation groups, Strocchi (1966) on complex coordinates in quantum mechanics, Miller (1972) for a representation of boson operators, and Lanczos (1968) in a complex representation of the Hamiltonian equations of motion. Now $(11) = 0$ gives the Cauchy–Riemann equations on the lhs and on the rhs the condition that D' have the symmetry defined by L^ζ. Thus, $(11) = 0$ states that the vacuum Hamiltonian as well as the vacuum symmetry of D' are determined in analytic regions of D'. The vacuum operators are now $\zeta_\mu \to \text{const} \times \tilde{u}_{\nu\mu}\, \partial/\partial x_\mu$ and the quantization rules (9) hold with $\alpha_\mu \to \alpha^\mu$. Noting that the operator

$$\alpha^\mu \zeta_\mu \sim \frac{\varepsilon}{2} \alpha^\nu \frac{\partial}{\partial \alpha^\nu} + \alpha^\rho \frac{\partial}{\partial \alpha^\rho} + i\,\alpha^\nu \frac{\partial}{\partial \alpha^\rho} - \alpha^\rho \frac{\partial}{\partial \alpha^\nu} \tag{13}$$

includes dilatation and rotation operators on the α planes, we find that TDT^{-1} is a *general* local transformation.

For $\phi = 0$, H' is properly defined on the momentum (operator) planes rather than configuration space. We therefore consider $H'(\partial_\mu, x_\mu)$ in momentum space. With $T(\alpha^\mu, \zeta_\mu) \to T(\Lambda^\mu, \eta_\mu)$ we obtain

$$\frac{d\eta_\mu}{d\Lambda^\nu} \frac{\partial H'(\eta_\mu, \Lambda^\mu)}{\partial \eta_\mu} = [\eta_\nu, H'(\eta_\mu, \Lambda^\mu)] + \frac{\partial H}{\partial \Lambda^\nu} \tag{14}$$

which reduces to

$$\tilde{a}_{\nu\mu} \frac{\partial H'(\eta_\mu, \Lambda^\mu)}{\partial \eta_\mu} = [\eta_\nu, H'(\eta_\mu, \Lambda^\mu)] \tag{15}$$

where $u \to a$ in momentum space. If $(15) = 0$, then momentum is conserved is a finite (analytic) region of momentum space. For momentum conservation in the entire momentum space we require also

$$\frac{\partial H'(\eta_\mu, \Lambda^\mu)}{\partial \eta_\nu} = [\Lambda^\nu, H'(\eta_\mu, \Lambda^\mu)] = 0 \tag{16}$$

In this case H' is a set of constants (operators). As will be shown, (16) cannot hold for the Dirac Hamiltonian with quantization rules such as (9) unless an auxiliary field is included in D'.

3. DIRAC HAMILTONIAN FOR A FREE FERMION WITH MASS

The left-handed symmetry of the Dirac Hamiltonian for a massless fermion is broken for finite mass, so that D is no longer polarized, but includes both left-handed (lh) and right-handed (rh) parts. Then D can be resolved into lh and rh polarizations plus longitudinal terms, and the Hamiltonian

$$D(\partial_\alpha) = \begin{vmatrix} -i\gamma^\mu\partial_\mu + M = X^\alpha\partial_\alpha \\ -i\partial_0 + M & & & -\partial_- \\ & -i\partial_0+M & -\partial_+ & \\ & \partial_- & i\partial_0+M & \\ \partial_+ & & & i\partial_0+M \end{vmatrix} \tag{17}$$

defined in Euclidean xyt space is a function of four complex variables $x_\alpha = [(x \pm iy), (M^{-1} \pm it)]$ and X^α are unitary matrices with elements ± 1. Equations (17) can be partitioned in the usual way into lh and rh operators on the $\psi_R = (\psi_1, \psi_3)$ and $\psi_L = (\psi_{2,4})$ spinors. The matrix (17) separates into a direct sum of $SU(2)$ matrices defined on the entire planes. Each includes lh and rh components. Under the transformation (12) we find $D(\partial_\alpha) \to D'(\zeta_\alpha)$ defined on transformed complex planes and $\partial_\alpha \to \zeta_\alpha$ defined on the right semiplanes. Thus $\zeta_\alpha = (\zeta_\alpha^+, \zeta_\alpha^-)$, where ζ_α^- is the complex conjugate of ζ_α^+. Then $\alpha^\alpha(x_\alpha^\pm)$ is also defined on the rh semiplanes. It is evident from

$$\frac{\partial D'(\zeta_\alpha)}{\partial \alpha^\beta} = \tilde{u}_{\beta\alpha}\frac{\partial D'(\zeta_\alpha)}{\partial x_\alpha} = [\zeta_\beta, D'(\zeta_\alpha)] = [\zeta_\beta, X^\alpha \tilde{v}_{\alpha\beta}\zeta_\beta] \tag{18}$$

that D' has no analytic regions on the x_α planes as $[\zeta_\beta, D'(\zeta_\alpha)] \neq 0$ and $\tilde{v} = \tilde{u}^{-1}$. From the Lie algebra L^ζ and the quantization rules from (9) it follows that the $\tilde{u}_{\alpha\beta}$ are constants. Another way to see that the $\tilde{u}_{\alpha\beta}$ are constants is to note that the α^β are infinitesimals $-\delta x^\beta$ and that $\partial\delta x/\partial x =$ const.

Since D is properly on the momentum (operator) planes we proceed as for equations (11)-(15) with $v_{\nu\mu}\zeta_\mu \to \eta_\alpha$ and $\alpha^\mu\zeta_\mu \to \Lambda^\alpha\eta_\alpha = X_\alpha\Lambda^\alpha X^\alpha\eta_\alpha$ with $[\Lambda^\alpha, X^\alpha] = 0$. Then

$$\frac{\partial D'(\eta_\alpha)}{\partial (X^\alpha\eta_\alpha)^\dagger} = X^\alpha\frac{\partial D'(\eta_\alpha)}{\partial \eta_\alpha} = 0 \tag{19}$$

In this form the momentum row (column) vectors in the disjoint $SU(2)$ matrices of D' are differentiated with respect to the conjugate column (row) momentum vectors. Then D' is analytic on the sum of the momentum planes and energy-momentum is conserved in a finite region. Also,

$$\frac{\partial D'(\eta_\alpha)}{\partial X^\alpha\eta_\alpha} = [X_\alpha\Lambda^\alpha, X^\alpha\Lambda_\alpha] \tag{20}$$

and since X^α has elements ± 1, then D' is not analytic on the entire space. Thus, for a free fermion with spin, momentum is conserved in a finite region of momentum space. The region is defined on the minor diagonal quadrants. We now ask the question: why is D' analytic in a finite region, or why is momentum not conserved on the entire planes? The physical basis is the two possible values of the spin momentum, from which it follows that the total momentum cannot be single-valued: this is evident also from the reduced representation of D', which is quadratic in the momentum. The function-theoretic answer can be stated as follows: according to Liouville's theorem, if D' is analytic on the entire planes, then D' is a matrix of constant elements, but D' does not necessarily commute with the vectors Λ^α. On the other hand, there are obvious examples (atoms, molecules) of systems that include lh and rh rotations and spins and within which momentum is conserved. In these systems the spins cancel in pairs. We consider in the next section a fermion to which is added an auxiliary field that compensates the fermion spin and assures momentum conservation and dynamical stability.

4. DIRAC HAMILTONIAN WITH SCALAR FIELD

In the previous section we found that the Hamiltonian of a free fermion with spin is not analytic on the entire momentum plane. In this section we add a scalar field $\phi(x_\alpha)$ and consider the analytic structure. It is well known that for a function of several variables analyticity establishes strong correlations among the variables. Thus, if $D[\partial_\alpha, \phi(x_\alpha)]$ is analytic, then the Cauchy-Riemann equations generate four partial differential equations in ϕ with solutions that are functions of the fermion momentum and spin, as may be demonstrated by solving the reduced matrix equations. Now consider a fermion with a scalar (Higgs) field ϕ. After transformation the Hamiltonian reads

$$D'(\zeta_\alpha, x_\alpha) = X^\alpha v_{\alpha\beta}\zeta_\beta - \lambda\phi(x_\alpha) \tag{21}$$

where $v_{\alpha\beta} = d\alpha^\beta / dx_\alpha$ and we have dropped the prime on ϕ; λ is the coupling and ϕ is quantized according to

$$[\phi_\alpha(x), \phi_\beta(y)] = c(x, y) \tag{22}$$

in the conventional notation. In analytic regions the terms in (21) are strongly correlated, so that

$$\frac{\partial D'(\zeta_\beta, x_\beta)}{\partial \alpha^\alpha} = \tilde{u}_{\alpha\beta}\frac{\partial D'(\zeta_\beta, x_\beta)}{\partial x_\beta} = [\zeta_\alpha, D'(\zeta_\beta, x_\beta)] = 0 \tag{23}$$

or

$$[\zeta_\alpha, X^\beta \tilde{v}_{\beta\alpha}\zeta_\alpha] = \lambda[\phi, X^\beta v_{\beta\alpha}\zeta_\alpha] \tag{24}$$

Thus, ϕ has the symmetry of G^{ξ}. This result is analogous to the degeneracy and cancellation effects in the supersymmetric vacuum. If D' is analytic on the entire x_α planes, then $D' = \text{const}$ and $\phi = \text{const}$. Thus, in configuration space analyticity on the entire planes leads to a special ϕ rather than a general result. In momentum space, on the other hand, analyticity on the entire planes represents a general conservation law. Equation (21) becomes in momentum space

$$D'(\eta_\alpha) = X^\alpha \eta_\alpha - \lambda\phi(\eta_\alpha) \tag{25}$$

Now D' has an analytic region as

$$\frac{d\eta_\alpha}{d\Lambda^\alpha}\frac{\partial D'(\eta_\alpha)}{\partial(X^\alpha\eta_\alpha)} = [X^\alpha\eta_\alpha, D'(\eta_\alpha)] = -\tilde{a}\lambda X^\alpha \frac{\partial\phi(\eta_\alpha)}{\partial\eta_\alpha} = 0 \tag{26}$$

As for configuration space, ϕ has the symmetry of the fermion momentum operators. However, the field cannot be analytic on the entire planes, since there is no overall momentum conservation law for ϕ in the presence of fermions. On the other hand, the Hamiltonian is now analytic on the entire planes if

$$\frac{\partial D'(\eta_\alpha)}{\partial(X^\alpha\eta_\alpha)} = [X_\alpha\Lambda^\alpha, (X^\alpha\eta_\alpha - \lambda\phi)] = 1 - \lambda X^\alpha \frac{\partial\phi}{\partial\eta_\alpha} = 0 \tag{27}$$

Since $[X_\alpha\Lambda^\alpha, X^\alpha\eta_\alpha] = \text{const}$, we see that ϕ represents a gauge function which cancels the nonanalytic part of the free fermion contribution to D'. Thus, we see that ϕ not only has the symmetry of the fermion, but also compensates the fermion momentum so that D' is constant. This result is familiar from supersymmetry and also from atomic physics; in the latter the total energy-momentum of an electron plus Coulomb field is conserved by compensation by the field of part of the electron momentum. Thus, D' is analytic on the entire planes for ϕ obtained from (26) and (27). The solution of (26) is $\phi(\eta_\alpha) = \phi_0$. The solutions of (27) are

$$\phi_\alpha(\eta_\alpha) = \phi_0 \exp X^\alpha \frac{d\eta_\alpha}{\lambda\phi(\eta_\alpha)} \tag{28}$$

According to (27), the field cannot be analytic on the entire planes, so the integral paths in (28) must lie partly within nonanalytic regions. In fact the multivalued contributions are the *only* contributions to the integral aside from a constant: if the paths enclose analytic regions, then

$$X^\alpha \frac{d\eta_\alpha}{\lambda\phi(\eta_\alpha)} = \lim_{\eta_\alpha \to \eta'_\alpha} \sum_{\eta_\alpha} \frac{X^\alpha}{\lambda} \frac{1}{\partial\phi(\eta_\alpha)/\partial\eta_\alpha} = 1 \tag{29}$$

and $\phi = \phi_0 \exp$.

The matrices in the exponent of (28) indicate that the ϕ_α do not commute. How is it that an initially spinless scalar field acquires spin dependence and a non-Abelian algebra? The answer is found in the vacuum constraint: for H analytic in entire momentum space, H is a constant; this is simply the complex variable expression of conservation of H in real space. It follows immediately that ϕ has the symmetry of the fermion and cancels the nonanalytic contribution of the fermion to H. The cancellation is a simple example of the well-known supersymmetric result for the vacuum H and can also be understood as the balancing of kinetic and potential energy contributions in the vacuum.

We now rewrite (28) so that

$$\phi_l(\eta_l) = \phi_0 \exp\Sigma^l \frac{d\eta_l}{\lambda\phi(\eta_l)} \tag{30}$$

where Σ^l are eight one-component matrices obtained by partitioning the two-component X^α matrices. The eight variables $\eta_l = [\pm(\eta_x \pm i\eta_y), \pm(\eta_0 \pm i\gamma)]$ are now associated one-to-one with the Σ^l, which generate an algebra L^Σ, and the operators $\Sigma^l \int d\eta_l/\lambda\phi$ generate the algebra $L^\Sigma \times L^\eta$. Writing s for the "space" components and t for the "time" components, one has for the Σ^l algebra

$$\begin{aligned} [\Sigma^t_+, \Sigma^t_-] &= 0 \\ [\Sigma^s_+, \Sigma^s_-] &= \pm(\Sigma^t_+ - \Sigma^t_-) \\ [(\Sigma^t_+ - \Sigma^t_-), \Sigma^s_\mp] &= \pm 2\Sigma^s_\mp \end{aligned} \tag{31}$$

where the algebraic $\pm$ signs belong to both disjoint subalgebras. Each disjoint chiral subgroup has a compact and a noncompact subgroup. The irreducible algebra is that of chiral $O(2,1)\times T_1 + O(2,1)\times T_1$, where T_1 is a diagonal in the Cartan subgroup. Noncompactness follows from the fact that ϕ is nonanalytic and a singularity exists. Each disjoint subgroup of G^Σ contains both lh and rh rotations and a nonanalytic singularity occurs in trying to go continuously from lh to rh rotations. Since there is a compact chiral subgroup of $G^\Sigma \sim O(3)\times O(3)$, it follows that spin interactions break the ϕ symmetry so that Abelian $G^\phi \to G^{\phi_l}$, where G^{ϕ_l} is non-Abelian and noncompact.

With $D' = \text{const}$, the integral in (21) becomes

$$\Sigma^l \frac{d\eta_l}{\lambda\phi(\eta_l)} = \ln \frac{-D' + \Sigma^l \eta_l}{\lambda\phi_0} \tag{32}$$

Integrating (30), we obtain

$$\phi_\alpha(x_\alpha, x_\alpha) = \phi_0 \exp \frac{-D'}{\lambda\phi_0} \frac{r_\alpha}{x_\alpha - im_\alpha^{-1}} \frac{r_\alpha}{x_\alpha - im_\alpha^{-1}} \tag{33}$$

where

$$m_\alpha^{-1} = \Sigma^\alpha / \lambda\phi_0 \tag{34}$$

and r_α are scale parameters. For a spinless fermion (33) gives the Jackiw-Nohl-Rebbi solution. Equation (33) has possible $1/r^2$-type singularities at points fixed by solutions of $(x_\alpha - im_\alpha^{-1})(x_\alpha - im_\alpha^{-1}) = 0$. However, the possible singularities of (33) do not correspond to simple poles removable by a gauge transformation (Atiyah and Ward, 1977), since $x_\alpha - im_\alpha^{-1}$ is a 4×4 matrix and the m_α are spin-dependent. Instead, the possible singularities are multivalued due to the nonanalytic character of ϕ. The denominator of (33) is a product of two fourth-order complex matrices, which gives an eighth-order polynomial in the x_α or in ϕ_0. The eight solutions fix scale lengths proportional to $|r_\alpha|$ in this unrenormalized result. Since scale lengths have the dimensions of mass^{-1}, the m_α can be identified with masses introduced by the interactions. In the renormalized result the possible divergences are removable, but the ϕ_α also acquire masses, provided the fermion has a finite mass. Thus, we conclude that no singularities exist, a result that also follows from supersymmetry.

5. VECTOR FIELD

The procedure of Section 4 can be carried over to vector interactions. The Dirac Hamiltonian

$$D(x_\mu) = -i\gamma^\mu[\partial_\mu + gA_\mu(x_\mu)] + M \tag{35}$$

becomes in momentum space

$$D'(\eta_\alpha) = X^\alpha[\eta_\alpha + gA_\alpha(\eta_\alpha)] \tag{36}$$

D' is analytic on the separate quadrants if

$$X^\alpha \frac{\partial D'(\eta_\alpha)}{\partial \eta_\alpha} = \frac{\partial A_\alpha(\eta_\alpha)}{\partial \eta_\alpha} = 0 \tag{37}$$

and D' is analytic on the entire planes if

$$\frac{\partial D'(\eta_\alpha)}{\partial \eta_\alpha} = X^\alpha + gX^\alpha \frac{\partial A_\alpha(\eta_\alpha)}{\partial \eta_\alpha} = 0 \tag{38}$$

A solution of (37) is $A_\alpha(\eta_\alpha)=A_\alpha^0$ and a solution of (38) is

$$A_\alpha(\eta_\alpha)=A_\alpha^0 \exp -\varepsilon \frac{d\eta_\alpha}{gA_\alpha(\eta_\alpha)}=0 \tag{39}$$

where ε is a constant. In the analytic region (39) becomes

$$\frac{1}{g}\frac{d\eta_\alpha}{A_\alpha(\eta_\alpha)}=\lim_{\eta\to\eta'}\sum_\eta \frac{1}{g}\frac{1}{\partial A_\alpha/\partial\eta_\alpha}=-1 \tag{40}$$

so that again multivalued contributions are the only contributions to the integral aside from a constant. When normalized to A_α^0 the gauge field in (39) is a "phase factor" defined on nonanalytic regions. Integrating (39), we obtain

$$A_\alpha^l(x_\alpha, x_\alpha)=A_\alpha^0 \exp \frac{\Sigma^l D'}{gA_l^0} \frac{r_\alpha}{x_\alpha+im_\alpha^{-1}} \frac{r_\alpha}{x_\alpha+im_\alpha^{-1}} \tag{41}$$

where

$$m_\alpha^{-1}=1/gA_\alpha^0 \tag{42}$$

and r_α are scale parameters. In (41) we again find spin components due to the interactions, but unlike the scalar case, the denominator has simple zeros at $x_\alpha=-im_\alpha^{-1}$ and $x_\alpha=-im_\alpha^{-1}$. Although the field (41) is nonanalytic everywhere, it is easily separable into a $1/x$ potential $A_\alpha^l(x_\alpha)\sim 1/(x_\alpha+im_\alpha^{-1})$ on the first quadrants for $im_\alpha^{-1}\neq 0$ and corresponding regions on the fourth quadrants.

The matrices in the exponential factor of A_α^l break the symmetry of A_α^0 so that if the A_α^0 have an Abelian algebra, then the A_α^l do not. By the Baker–Hausdorff theorem the A_α^l have a non-Abelian symmetry, since the Σ^l do not commute. The interactions are responsible for the symmetry breaking in the same way as for the scalar field, and they generate additional components $A_\alpha \to A_\alpha^l$, $l=1, 2, \ldots$, as for the scalar field. For the vector field, however, the l indices represent directions fixed by spin matrices in an $\exp(\Sigma^l D'/gA_l^0)$ color space. There are four space-time parameters x_α and x_α and as $\alpha\sim l$ there are four corresponding color and scale parameters. Then A_α^l has eight components defined on the minor diagonal quadrants and A_l^α has eight components defined on the major diagonal quadrants. The overall symmetry is $SU(3)\times SU(3)$.

6. SUMMARY AND DISCUSSION

The formalism of symmetry groups, conservation laws, and selection rules is of fundamental importance in quantum field theory and particle theory. In the complex domain these concepts are related to analytic regions and functions. I have shown that a continuous transformation of the Hamiltonian $H(z_\mu)$ generates Cauchy–Riemann equations which define analytic regions of H on the z_μ planes. The solutions of the Cauchy-Riemann equations are vacuum states and yield the vacuum Hamiltonian. The conserved functions and ground states in a real variable representation go over to analytic functions and analytic regions in a complex representation. This formalism has been used to obtain vacuum states and Hamiltonians in the following cases: (1) a free fermion, (2) a fermion with a scalar field, and (3) a fermion with a vector field. It is found that the Hamiltonian of a free fermion is not analytic in entire momentum space. By adding an auxiliary field ϕ, the Hamiltonian can become analytic in the entire space and the field effectively cancels the nonanalytic part of the free fermion Hamiltonian. The field then appears as a "phase factor" integral defined on nonanalytic regions in momentum space. The contributions from the integral are spin-dependent and multivalued except at the analytic point $\phi = \phi_0$. Otherwise, $\phi \to \phi_l$ and the ϕ_l have a non-Abelian algebra. We ask the question: how is it that an initially spinless scalar field can acquire spin components and a non-Abelian algebra? The answer is found in the vacuum constraint on the supersymmetric Hamiltonian: for H analytic in the entire momentum space, H is a constant. Then ϕ has the symmetry of the fermion and cancels the nonanalytic contribution of the fermion to the vacuum Hamiltonian. This type of result is familiar from supersymmetry and can be understood as partial cancellation of the kinetic and potential energy contributions to the vacuum Hamiltonian. In configuration space ϕ has possible $1/r^2$-type nonanalytic singularities, which cannot be removed by a gauge transformation. The singularities fix quadric and higher order surfaces on which ϕ may have divergences. The multivaluedness of the field and the acquisition of spin components follow from the Dirac equation itself: in the completely reduced representation there are four solutions for each chirality eigenvalue of the fermion. Thus, $\phi(x_\mu) \to \phi_l(x_\mu)$, $l = 1, \ldots, 8$, and the ϕ_l have a non-Abelian algebra. The Jackiw–Nohl–Rebbi solution is obtained for a spinless fermion and the Coulomb-type $1/x$ solution follows.

For a vector field the results are analogous, except that the coordinates are single-valued at the singularities and independent of spin. Nevertheless, A_μ is multivalued and also has four solutions for each chirality eigenvalue of the fermion. Thus, as with the scalar field, the vector field acquires spin components by interaction with the fermion in the vacuum and $A_\mu \to A_\mu^l$, $l = 1, 2, \ldots, 8$. The interactions break the symmetry of A_μ and the A_μ^l algebra is non-Abelian. The l indices represent directions fixed by spin matrices in a spin- and scale-dependent color space. The octet A_μ^l is defined on the minor diagonal quadrants and the A_l^{μ} octet is defined on the major diagonal quadrants. The overall vector field symmetry is $SU(3)\times SU(3)$.

We now turn to the occurrence of the Jackiw-Nohl-Rebbi and Coulomb-type solutions for spinless fermions. As is well known, both are solutions of the Laplace equation. How is it that these solutions appear in the integral transformation of $\phi(\eta_\alpha)$ to configuration space? The answer is found in the reducibility of the linear Dirac equation. The irreducible Hamiltonian $\mathscr{H}$ is fourth-order in ϕ and second-order in $\zeta^\alpha\zeta_\alpha$ and includes terms $\zeta^\alpha\phi\zeta_\alpha\phi$. If the vacuum Lagrangian $\mathscr{L}$ is constructed from $\mathscr{H}$ for $M=0$ and $\zeta_\alpha=\partial_\alpha$, we find

$$\mathscr{L}(\phi_\alpha,\phi_\beta)=\partial_\alpha\phi_\alpha\partial_\alpha\phi_\alpha+a\phi_\alpha\phi_\beta+b\phi_\alpha\phi_\beta\phi_\alpha\phi_\beta+\text{spin terms} \tag{43}$$

which yields the Laplace equation for $a=b=0$ or for

$$a\phi_\beta+b(\phi_\beta\phi_\alpha\phi_\beta+\phi_\alpha\phi_\beta\phi_\beta)=0 \tag{44}$$

Otherwise, the equation of motion has kink solutions.

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